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Convergence to the Ground-State Energy in the Thermodynamic Limit of the Ising Model in a Strong Transverse Field

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For the quantum mechanical Ising model in a strong transverse field we show that the convergence of the ground-state energy per site as the volume goes to infinity has an Ornstein–Zernicke behavior. That is, if the diameter of the d-dimensional lattice is given by L, the absolute value of the difference of the ground-state energy per site and its limit is asymptotically $\exp(-\xi L)\, L^{-d/2}$ for some positive constant ξ. We also show that the correlation function has the same behavior. Our results are derived by cluster expansions, using a method of Bricmont and Fröhlich which we extend to the quantum mechanical case.

KEY WORDS: Quantum mechanical spin systems; polymer expansions; finite-volume energy.

1. INTRODUCTION

The model considered in this paper is the Ising model in a strong transverse field on a d-dimensional cubic lattice. The main result concerns the convergence of the finite-volume ground-state energy per site to its infinite-volume limit. The new result which we prove here is a statement about the rate of convergence—namely, that the convergence is exponential with a specific power-law correction, $e^{-\xi L}/L^{d/2}$, where L is the "diameter" of the lattice and $\xi > 0$. Although the model treated in this paper is well understood, as this result is the first rigorous proof of the Ornstein–Zernicke law regarding the finite-volume energy, the result may yet be useful for numerical studies of the correlation length of this model because the numerics are usually limited to rather small lattice sizes. Whether the method may be

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extended to other less well-known models is an open question. After getting the result on the convergence of the ground-state energy it will also be shown that the same asymptotic behavior is true for the space correlation functions. That is, the correlation function $\langle \sigma_0^x \sigma_{le_j}^x \rangle$ (where e_j is one of the unit axes in the lattice and l is an integer) is asymptotically equal to $e^{-\xi l}/l^{d/2}$ for the same value of ξ as above. The Ornstein–Zernicke decay of the correlation functions for this model was found earlier by Kennedy.[4]

The results are obtained by a series of cluster expansions of the free energy per site at finite volume and finite β (where β is the inverse temperature), the first of which is standard, and the second being an application of a method developed by Bricmont and Fröhlich[1] for classical systems to the quantum mechanical system. The first expansion uses the standard method of writing a quantum mechanical spin system as a classical system in one higher dimension, as first rigorously done in ref. 3. The techniques of Bricmont and Fröhlich are used on the results of the first expansion by rewriting the first expansion as a gas of *one-dimensional* interacting rods. The quantity of interest in this paper is the difference of the free energy per site in finite volume and the infinite-volume limit of the free energy per site. To find the ground-state energy, we take the zero-temperature limit of the expansion of the free energy. The infinite-volume limit follows the zero-temperature limit. This method yields a result for the given difference whose asymptotic behavior can be extracted by standard methods.

2. MAIN RESULTS

2.1. Statement of the Theorem

In this paper we will consider the Ising model in a strong transverse field. We define the model by giving the Hamiltonian of our system as well as the Hilbert space on which the Hamiltonian acts. The Hilbert space is the tensor product space of $|\Lambda|$ copies of $\mathbf{C}^2$, and the Hamiltonian acting on this Hilbert space is

$$H_\Lambda = -\varepsilon \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z + \sum_i (\sigma_i^x + 1) \tag{1}$$

where the sum is over nearest-neighbor pairs of sites $\langle ij \rangle$ on the lattice Λ with periodic boundary conditions, and the σ's are the usual Pauli spin operators. Throughout this paper the lattice Λ will be $\{1,..., L\}^d$. We will develop an expansion for the free energy of this system in the ground state and intend to analyze its convergence asymptotically as $L \to \infty$. As a

preliminary step we apply a unitary transformation which rotates every site about the y axis by $\pi/2$ radians. After this transformation (1) becomes

$$H_\Lambda = -\varepsilon \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x - \sum_i (\sigma_i^z - 1) \tag{2}$$

Since Λ is completely determined by L, we will designate H_Λ by H_L hereafter.

The free energy per site is defined by

$$f_L(\beta) \stackrel{\text{def}}{=} -\frac{1}{\beta L^d} \ln Z_L(\beta) \tag{3}$$

with

$$Z_L(\beta) \stackrel{\text{def}}{=} \operatorname{Tr}[\exp(-\beta H_L)] \tag{4}$$

where the sum is over all configurations of the spins on the lattice. The infinite-volume limit of the function f_L is of interest here, and we will denote the limit (which we will show exists) by f_∞.

The theorem which we will prove in this paper follows.

Theorem 1. For the Hamiltonian given in (1) in dimensions $d \geqslant 1$, for sufficiently small ε, there are functions f_L and f_∞ which depend on d and ε such that $f_L(\beta)$ [defined in (3)] approaches f_L as $\beta \to \infty$, and, in turn, f_L approaches f_∞ as $L \to \infty$. There exists a $\xi > 0$, which depends only on d and ε, such that the convergence of f_L is asymptotically given by

$$|f_\infty - f_L| \sim \exp(-\xi L)\, L^{-d/2} \tag{5}$$

as $L \to \infty$. The quantity on the left-hand side is asymptotic to that on the right-hand side in the sense that there are strictly positive constants p and $q < \infty$ such that the left-hand side is bounded below (respectively, above) by p (q) times the right-hand side.

The existence of the limits f_L and f_∞ is well known, and can be proved by general methods.[2] What is new in this theorem is the asymptotic behavior of the approach to the limit.

2.2. The First Expansion

Using a slight variant of the Trotter product formula, we begin the first expansion of Z_L:

$$Z_L(\beta) = \lim_{N \to \infty} \operatorname{Tr} \left\{ \left[\exp\left(\frac{1}{N} \sum_i (\sigma_i^z - 1) \right) \left(1 + \frac{\varepsilon}{N} \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x \right) \right]^{N\beta} \right\} \tag{6}$$

We choose the usual complete set of basis vectors $\{|\Psi\rangle\}$, where $\sigma_i^z|\Psi\rangle = \sigma_i|\Psi\rangle$ with $\sigma_i = \pm 1$ (if $\sigma_i = +1$, the spin at site i is "up"; if $\sigma_i = -1$, the spin at site i is "down") for each $i \in \Lambda$. By inserting a sum over the set of basis vectors between each of the $N\beta$ factors in the product, the right-hand side of (6) becomes

$$\sum_{\Psi_0,\ldots,\Psi_{\beta-(1/N)}} \prod_{t=0}^{\beta-(1/N)} \langle \Psi_t | \exp\left[\frac{1}{N}\sum_i (\sigma_i^z - 1)\right]\left(1 + \frac{\varepsilon}{N}\sum_{\langle ij\rangle} \sigma_i^x \sigma_j^x\right) |\Psi_{t+(1/N)}\rangle \tag{7}$$

with $|\Psi_\beta\rangle \stackrel{\text{def}}{=} |\Psi_0\rangle$. The exponential terms are diagonal in the basis, so we now have

$$\sum_{\Psi_0,\ldots,\Psi_{\beta-(1/N)}} \prod_{t=0}^{\beta-(1/N)} \exp\left\{\frac{1}{N}\sum_i [\sigma_i(t) - 1]\right\} \langle \Psi_t | 1 + \frac{\varepsilon}{N}\sum_{\langle ij\rangle} \sigma_i^x \sigma_j^x |\Psi_{t+(1/N)}\rangle \tag{8}$$

where $\sigma_i(t)$ is the value of the spin at site i at time t.

We will denote by Ξ the time axis ($= \{0, 1/N, \ldots, \beta - 1/N\}$). Each term in the sum (8) has a geometrical interpretation in terms of bonds in $\Lambda \times \Xi$. The ground state of the unperturbed Hamiltonian has all of the spins in the "up" direction. If we consider only those spins in the "down" direction (i.e., those not in the ground state), then we can describe the terms in the sum by the "paths" of these spins in $\Lambda \times \Xi$. If spin i is out of the ground state at time t, then it will appear in (8) with a factor of $\exp(-2/N)$, and we will associate the bond from (i, t) to $(i, t + 1/N)$ with that spin. When a $\sigma_i^x \sigma_j^x$ term flips a pair of nearest-neighbor spins, then we will associate with the bond connecting the two sites a weight of ε/N in (8). Thus, every term in (8) corresponds to a set of bonds in $\Lambda \times \Xi$. It is easily seen that each such set of bonds has no boundary, or, equivalently, every site must touch an even number of these bonds. Bonds parallel to the Ξ axis we call *time bonds*, while those connecting sites in Λ we call *space bonds*.

We can now write (8) as a sum over sets of bonds in $\Lambda \times \Xi$. The sets of bonds are partitioned into connected components called *contours*. As mentioned above, contours have no boundary, and thus, under periodic boundary conditions of $\Lambda \times \Xi$, all contours are closed paths, or loops. When we rewrite Z_L as a sum over contours there will be the constraint that there can be at most one space bond at each t value. The constraint on the sum follows from the following observation: the off-diagonal factors in (8) (i.e., those associated with the space bonds) are given by

$$\langle \Psi_t | \, 1 + \frac{\varepsilon}{N} \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x \, | \Psi_{t+1/N} \rangle$$

$$= \begin{cases} 1 & \text{if} \quad \Psi_t = \Psi_{t+1/N} \\ \dfrac{\varepsilon}{N} & \text{if} \quad \langle \Psi_t | \, \sigma_i^x \sigma_j^x \, | \Psi_{t+1/N} \rangle = 1 \quad \text{for a single} \quad \langle ij \rangle \in \Lambda^* \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where Λ^* indicates the set of bonds of the lattice Λ.

We therefore associate to each contour ω a *weight* $W_L(\omega)$ composed of a factor of $\exp(-2/N)$ for every time bond in ω, and a factor of ε/N for every space bond in ω. In terms of contours and their weights, (8) is now a weighted sum over sets of contours in $\Lambda \times \Xi$:

$$Z_L(\beta) = \lim_{N \to \infty} \tilde{\sum_{\Omega = \{\omega_1, \ldots, \omega_n\}}} W_L(\Omega) \tag{10}$$

where $\tilde{\sum}$ is a sum over allowed disjoint sets of contours in $\Lambda \times \Xi$, and $W_L(\Omega)$ is the product of the weights of the contours in Ω. Aside from the requirement that all contours have no boundary, the allowed sets of contours which occur in the sum are those for which at most one space bond exists at each t value. The magnitude of the difference of the sum with the constraint (of at most one space bond at each time) and that without the constraint vanishes as N increases,[4] and thus we may replace the constrained sum with an unconstrained sum.

To proceed, we need to show that an expansion exists for the logarithm of (10) in the zero-temperature limit. This can be done by the application of a result by Kotecky and Preiss,[5] in which they prove that there is a convergent polymer expansion if the following condition holds for the weights of the polymers $W_L(\omega)$. Let $S(\omega)$ be the number of space bonds in ω, and let $T(\omega)$ be the number of time bonds in ω. In terms of (10) the condition which must be satisfied is that for a constant $\rho < 1$, for each contour ω_0,

$$\sum_{\omega : \omega_0 \cap \omega \neq \varnothing} W_L(\omega) \, e^{|\omega|} \leqslant \rho \, |\omega_0| \tag{11}$$

where $|\omega| = S(\omega) + T(\omega)/N$. This condition has been proved to hold for the model under consideration by Schorr[6] and Kennedy.[4] The expression for $|\omega|$ is chosen for the following reasons. Because every contour is a loop, there are an equal number of space bonds and time segments (i.e., connected time bonds uninterrupted by space bonds). Because of this, we can "transfer" the $1/N$ factor from each space bond to one of the time segments,

allowing us to assign a weight of ε to each space bond, and a weight of $\exp[-2(\text{length})/N]/N$ to each time segment.

We therefore have a convergent polymer expansion which can be written

$$\ln Z_L(\beta) = \lim_{N \to \infty} \sum_{\Omega} \phi(\Omega) \tag{12}$$

where Ω is a set of connected contours or a *polymer*, and $\phi(\Omega)$ is the product of the weights of the component contours multiplied by the connected part of the potential. If we let

$$f_\infty = \lim_{L \to \infty} \lim_{\beta \to \infty} \frac{1}{\beta L^d} \ln Z_L(\beta)$$

then we must now consider the difference $f_\infty - f_L$, where f_L is defined in (3). In order to simplify the notation, we will take the limits to be implicit where necessary, and refer back to them only when required. Any expression for f_∞ must fix the location of the polymers, and hence we define the *corners* of a polymer to be the points on the polymer where a time segment is connected to a space bond. Any polymer which has nonzero weight in the limit has at least four corners, and so if we order the sites of the lattice lexicographically, we are able to define $\mathscr{C}_0(\Omega)$ to be the first corner in Ω. We write the two quantities in the following form:

$$f_L = \sum_{\Omega = \{\omega_1, \ldots, \omega_n\}} \frac{1}{\beta L^d} V_L^C(\Omega)\, W_L(\omega_1) \cdots W_L(\omega_n) \tag{13}$$

and

$$f_\infty = \sum_{\substack{\Omega = \{\omega_1, \ldots, \omega_n\} \\ \mathscr{C}_0(\Omega) = 0}} V_\infty^C(\Omega)\, W(\omega_1) \cdots W(\omega_n) \tag{14}$$

Note that the connected parts of the potential in (13) and (14) are not the same functions.

At this point the lattice is still $\Lambda \times \Xi$, but in order to simplify things somewhat in the following we will do the calculations as if the $\beta \to \infty$ limit has already been completed. However, since the $\beta \to \infty$ limit is done before the $L \to \infty$ limit, it is easy to check in the following that the convergence of f_L to f_∞ is unaffected by this change. With that in mind, the difference of the two f_L and f_∞ consists of the terms which appear in only one of either the infinite volume or the given finite volume. Let $\mathscr{L}_L$ be the polymers which have nonzero weight in the finite lattice and zero weight in

the infinite lattice. Also let $\mathscr{L}_\infty$ be the polymers which have nonzero weight in the infinite lattice and zero weight in the finite lattice. Now

$$|f_\infty - f_L| = \sum_{\Omega = \{\omega_1, \ldots, \omega_n\}} \widehat{V^C}(\Omega)\, \hat{W}(\Omega) \tag{15}$$

where

$$\widehat{V^C}(\Omega) = \begin{cases} V_\infty^C(\Omega)\, \delta_{\mathscr{C}_0(\Omega),0} & \text{if} \quad \Omega \in \mathscr{L}_\infty \\ (1/L^d)\, V_L^C(\Omega) & \text{if} \quad \Omega \in \mathscr{L}_L \end{cases} \tag{16}$$

and

$$\hat{W}(\Omega) = \begin{cases} \prod_i W(\omega_i) & \text{if} \quad \Omega \in \mathscr{L}_\infty \\ \prod_i W_L(\omega_i) & \text{if} \quad \Omega \in \mathscr{L}_L \end{cases} \tag{17}$$

In order to extract information about the asymptotics of the convergence of f_L to f_∞, we must be able to estimate the quantity in (15). We will be able to calculate the asymptotic behavior of the right-hand side of (15) as $L \to \infty$, and then we will have proved the theorem.

We next determine which polymers are in the sum in (15). First, the polymers which $\mathscr{L}_\infty$ contributes to the sum are those which are not valid polymers in the lattice of size L^d, but which are valid polymers in lattices of all sizes at least L_0^d for some $L_0 > L$. It is important to note that this set of polymers is included in the set of polymers Ω for which $\operatorname{diam}(\Pi_\Lambda(\Omega)) > L$, where $\Pi_\Lambda(\Omega)$ is the projection of Ω onto Λ. Second, the only polymers which $\mathscr{L}_L$ contributes to the sum are those which, in the periodic extension of Λ, are infinite paths (i.e., polymers which wrap around Λ). We will call the polymers which have this property *ribbons*. Note that there are also contours which are ribbons; when we need to differentiate between the two we use the terms ribbon polymer or ribbon contour. The asymptotic behavior of the sum (15) will be determined by those polymers which have the largest weights as $L \to \infty$, and therefore we must now estimate the relative weights of the two sets delineated above.

When $\Omega \in \mathscr{L}_\infty$ we can find an upper bound for the sum of weights of these polymers by finding an upper bound for the sum of weights over the polymers whose projection in Λ has a diameter greater than L. Thus, we will use the following bound:

$$\sum_{\Omega : \Omega \in \mathscr{L}_\infty} \widehat{V^C}(\Omega)\, \hat{W}(\Omega) \leqslant \sum_{\substack{\Omega : \mathscr{C}_0(\Omega) = O \\ \Omega \in D_L}} V_\infty^C(\Omega) \prod_i W(\omega_i) \tag{18}$$

where $D_L = \{\Omega \,|\, \operatorname{diam}(\Pi_\Lambda(\Omega)) > L\}$. The right-hand side of the preceding can be bounded as follows. Since every polymer in the sum belongs to D_L,

there are at least $2L$ space bonds in every Ω. With each of these space bonds there is a factor of ε which appears in the weight of the polymer for a total weight of, at most, ε^{2L}. The final bound which we find is

$$\sum_{\Omega:\Omega\cap\mathscr{L}_\infty\neq\varnothing} \hat{V}^C(\Omega)\,\hat{W}(\Omega)\leqslant O(\varepsilon^L e^{-\mu L}) \tag{19}$$

where μ can be made as large as desired by taking sufficiently small ε.

When $\Omega\in\mathscr{L}_L$, the polymer is a ribbon. Let p_i be the winding number in the ith direction of a given ribbon. Given the winding numbers of a ribbon, it is easy to see that a ribbon has at least $L(|p_1|+\cdots+|p_d|)$ space bonds. However, if one of the contours in a ribbon polymer is not a ribbon itself, then the polymer must have at least $2L(|p_1|+\cdots+|p_d|)$ space bonds. The ribbons which include exactly one ribbon contour and have only a single $|p_i|=1$ and the other p_i equal to zero will be the leading-order contributions to the sum (15), while all the other ribbons will have higher-order contributions which we will simply bound. Let $\mathscr{R}_2$ be the set of ribbon polymers which do not include exactly one ribbon contour or which have $\sum_i|p_i|\geqslant 2$. From the discussion above it is clear that these ribbons will have at least $2L$ space bonds. The bound on the sum over polymers in $\mathscr{R}_2$ can then be found in a way similar to the bound (19), which yields

$$\sum_{\Omega:\Omega\cap\mathscr{R}_2\neq\varnothing} \hat{V}^C(\Omega)\,\hat{W}(\Omega)\leqslant O(\varepsilon^L e^{-\mu L}) \tag{20}$$

2.3. The Second Expansion

The only terms from (15) which have not yet been considered are ribbons in $\mathscr{L}_L\backslash\mathscr{R}_2$, which are those terms contributing to the leading-order behavior of the sum. The only polymers in $\mathscr{L}_L\backslash\mathscr{R}_2$ are those which include exactly one ribbon contour which winds around Λ in only one direction. Our next step is to consider the lowest-energy polymers which are in this group. We will find that the polymers in this group can be characterized by their projections on the single axis in the direction of their sole nonzero winding number. Following ref. 1, we will develop a new expansion in a single dimension whose asymptotic behavior as $L\to\infty$ can be found.

The polymers which have not yet been included in either of the bounded sums (19) or (20) will have one ribbon contour with $|p_i|=\delta_{ij}$ for some j. Let $\mathscr{R}_1^j$ be the set of ribbon contours for which $|p_i|=\delta_{ij}$. Let $P_j(r)$ denote the coordinates of a point in the projection of Λ onto the hyperplane orthogonal to e_j, and let r_j denote the jth coordinate of a point in Λ. Also let $(P_j(r), r_j, t)$ denote the full set of coordinates of a point in $\Lambda\times\Xi$. Each

of the ribbon contours in $\mathscr{R}_1^j$ goes from $(P_j(r), 0, t)$ to $(P_j(r), L, t)$ for some value of $P_j(r)$ and t. We can fix the location of the polymers if we always make $P_j(r)=0$ and $t=0$. Geometrically, a polymer which has one of these ribbon contours will appear as a path with other contours overlapping with it (see Fig. 1, which depicts a single ribbon contour only). The lowest weight which such a polymer can have occurs when the ribbon contour goes straight across the lattice. However, in the quantum mechanical case, the time direction is continuous, and there is a zero probability that the ribbon does not change its Ξ value at every lattice site which it touches. The typical ribbon contour will therefore have some nonzero time segment at every such lattice site. Thus the lowest-energy polymers must be those which have single ribbon contours whose projections onto Λ go straight across the lattice (i.e., without any back- or sidetracking), and the excitations are those portions of the contour in which back- or sidetracking occurs. Let $\Pi_i(\Omega)$ be the projection of any polymer Ω onto the ith axis in Λ (which will be denoted by e_i hereafter). Consider a fixed interval $X=(m, m+1]$ in $\Pi_i(\Omega)$. If the set $\Pi_i^{-1}(X)\cap\Omega$ contains more than one space bond, then we will say that X is in the set of *excitation intervals* of Ω.

Let $\omega_0\in\Omega$ be the $\mathscr{R}_1^j$-ribbon in the polymer Ω. For any interval X in e_j, define

$$K(X, \omega_0)=\sum_{\substack{\Omega:\Omega\cap\omega_0\neq\varnothing \\ \Pi_i(\Omega)=X}} -\phi(\Omega) \tag{21}$$

where $\phi(\Omega)$ is the same function defined in (12). Because $\Pi_i(\Omega)=X$, there are at least $2|X|$ space bonds in Ω, and thus it is easy to see that $K(\cdot, \omega_0)$ is bounded by

$$|K(X, \omega_0)|\leqslant\varepsilon^{|X|}\exp(-\mu|X|)\,|\omega_0\cap\Pi_i^{-1}(X)| \tag{22}$$

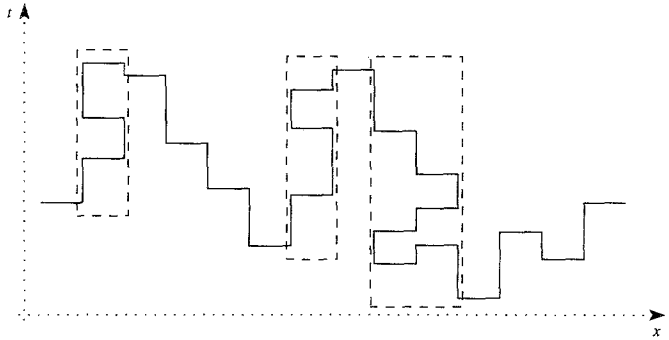

Fig. 1. A typical ribbon contour in $d=1$. The excitations are enclosed by the shaded boxes. Because of the periodicity of the lattice, the left and right ends of the ribbon are actually the same point.

where $\mu > 0$ can be made as large as desired by taking ε sufficiently small. Since $K(X, \omega_0)$ is small, we define $k(X, \omega_0)$ by

$$\exp K(X, \omega_0) = 1 + k(X, \omega_0) \tag{23}$$

Those elements of ω_0 for which have their projections in the excitation intervals of ω_0 we call the *excitations* of ω_0, and we denote the excitations of ω_0 by $E(\omega_0)$. Let the terms in the sum on the right-hand side of (15) which contain exactly one ribbon contour in $\mathscr{R}_1^j$ (for some j) be denoted $f_{\mathscr{R}_1}^L$. In terms of these new definitions, $f_{\mathscr{R}_1}^L$ is

$$\begin{aligned} f_{\mathscr{R}_1}^L &= \sum_{j=1}^{d} \sum_{\omega_0 \in \mathscr{R}_1^j} W(\omega_0) \prod_{X \subset e_j} [1 + k(X, \omega_0)] \\ &= \sum_{j=1}^{d} \sum_{\omega_0 \in \mathscr{R}_1^j} W(\omega_0) \sum_{X_1, \ldots, X_n} \prod_i k(X_i, \omega_0) \\ &= \sum_{j=1}^{d} \sum_{Y \subset e_j} \sum_{\substack{\omega_0 \in \mathscr{R}_1^j: \\ \Pi_j(E(\omega_0)) = Y}} W(\omega_0) \sum_{X_1, \ldots, X_n} \prod_i k(X_i, \omega_0) \end{aligned} \tag{24}$$

where the sum over $X_1, \ldots, X_n$ is over distinct intervals which can overlap, and the sum over Y is over sets of disjoint intervals. Note that those parts of $\omega_0 \in \mathscr{R}_1^j$ which are not excitations consist of only space bonds parallel to the e_j axis and one time segment for each space bond. Because there are equal numbers of time segments and space bonds, we can form a one-to-one pairing of each space bond with exactly one of the time segments touching it. Let t_i be the length of the time segment associated with the ith bond not in $E(\omega_0)$. A single excitation of ω_0 is the part of ω_0 in $\Pi_j^{-1}(Y_i)$ for a fixed i. An excitation of $\omega_0 \in \mathscr{R}_1^j$ must have a single space bond parallel to e_j entering it from the left and a single space bond parallel to e_j exiting it from the right. For the ith excitation, denote the coordinates projected onto the d-dimensional hyperplane orthogonal to e_j of the space bond entering the excitation by $(\mathbf{r}_i^-, s_i^-)$, and of the space bond exiting the excitation by $(\mathbf{r}_i^+, s_i^+)$. For each excitation interval define $(\mathbf{r}_i, s_i) = (\mathbf{r}_i^+, s_i^+) - (\mathbf{r}_i^-, s_i^-)$. Because of the periodic boundary conditions we have the constraint $\delta(\sum_i \mathbf{r}_i) = 1$ and $\delta(\sum_i s_i + \sum_i t_i) = 1$, where

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

We will soon need to sum the weights of the time segments over all lengths. Recall that a time segment has an overall factor of $1/N$, and a factor of $\exp(-2/N)$ for each time bond. Thus,

$$\sum_{i=-\infty}^{\infty} \frac{1}{N} \exp\left(\frac{-2|i|}{N}\right) = \frac{1}{N} + 2\sum_{i=1}^{\infty} \frac{1}{N} \exp\left(\frac{-2i}{N}\right)$$
$$= \frac{1}{N} + \frac{2}{N}\left[\exp\left(\frac{2}{N}\right) - 1\right]^{-1} \tag{26}$$

and in the limit $N \to \infty$ the sum converges to 1. The polymer with the lowest weight is a single ribbon without any excitations. Such a polymer simply has L space bonds in it (each with a weight of ε), and the sum over all lengths of its time segments converges to 1, giving it a total weight of ε^L. If we factor the weight of these polymers out of the sum, then we must put back in a factor of $\varepsilon^{-|Y|}$ for every excitation Y, while the weight of ω_0 outside of all excitations cancels the remaining ε^{-1} factors. Now (24) can be rewritten as

$$f^L_{\mathscr{R}_1} = \varepsilon^L \sum_{j=1}^{d} \sum_{Y_1,\ldots,Y_m} \sum_{\substack{\omega_0: \\ \Pi_j(E(\omega_0)) = \bigcup_i Y_i}} \sum_{X_1,\ldots,X_n} \prod_i W_t(i)$$
$$\times \prod_g \varepsilon^{-|Y_g|} U(\Pi_j^{-1}(Y_g) \cap \omega_0) \prod_h k(X_h, \omega_0) \tag{27}$$

where the Y_i are disjoint intervals (in e_j), the X_i are distinct intervals [as in (24)], $W_t(i)$ is the weight of the time segment associated with the ith space bond, and $U(\cdot)$ is the weight of the argument's space bonds only. After describing ω_0 in terms of the $(\mathbf{r}_i, s_i)$ and t_i we get the equation

$$f^L_{\mathscr{R}_1} = \varepsilon^L \sum_{j=1}^{d} \sum_{Y_1,\ldots,Y_m} \sum_{\{(\mathbf{r}_i, s_i)\}, \{t_i\}} \delta\left(\sum_i \mathbf{r}_i\right) \delta\left(\sum_i s_i + \sum_i t_i\right)$$
$$\times \sum_{\substack{\omega_0: (\mathbf{r}, s), t \\ \Pi_j(E(\omega_0)) = \bigcup_i Y_i}} \sum_{X_1,\ldots,X_n} \prod_i W_t(i) \prod_g \varepsilon^{-|Y_g|} U(\Pi_j^{-1}(Y_g) \cap \omega_0)$$
$$\times \prod_h k(X_h, \omega_0) \tag{28}$$

where the sum over ω_0: $(\mathbf{r}, s)$, t means a sum over all ω_0's which conform to the given $(\mathbf{r}^{\pm}, s^{\pm})$ and t restrictions. The symmetry of the d axes in Λ allows us to replace the sum over the axes in (28) by fixing the value of j and multiplying by d.

The union of all intervals in the above equation can be partitioned into disjoint intervals $I_1,\ldots, I_m$. The weights of these intervals are now

$$\mathscr{U}(I, \mathbf{r}, s) = \sum_{X_1,\ldots,X_d} \sum_{\omega_0} \prod_j k(X_j, \omega_0) \prod_{i \in \Pi_j^{-1}(I)} W_t(i)\, \varepsilon^{-|\Pi_j(E(\omega_0))|} U(E(\omega_0)) \tag{29}$$

where the sums are over ω_0 and the X_i with $\Pi_j(E(\omega_0)) \cup (\bigcup_i X_i) = I$, the X_i distinct, ω_0 entering the interval at 0, and exiting the interval at $(P_j(r), |I|, s)$. With this definition, (28) becomes

$$f^L_{\mathscr{R}_1} = \varepsilon^L d \sum_{\substack{I_1,\ldots,I_m: \\ \text{disjoint}}} \sum_{(\mathbf{r}_1,s_1),\ldots,(\mathbf{r}_m,s_m)} \sum_{t_1,\ldots,t_m} \prod_i W(t_i)$$
$$\times \delta\left(\sum_i \mathbf{r}_i\right) \delta\left(\sum_i s_i + \sum_i t_i\right) \prod_i \mathscr{U}(I_i, \mathbf{r}_i, s_i) \tag{30}$$

We next discuss bounds on $|\mathscr{U}(I, \mathbf{r}, s)|$. From (22) we see that each factor of $k(X, \omega_0)$ in (29) is no larger than $\varepsilon^{|X|} \exp(-\mu |X|)$ times a term proportional to the size of that part of ω_0 inside of $\Pi_j^{-1}(X)$. Because ω_0 extends across a distance of at least $|s|$ in the Ξ direction, it contributes a factor of $\exp(-2|s|/N)$. Of this factor, $\exp(-|s|/N)$ can be factored out of the sum, and the remaining $\exp(-|s|/N)$ is still sufficient to control the sum. The number of space bonds which ω_0 contains can be seen to be at least $|\mathbf{r}|$ bonds (where $|\mathbf{r}|$ is the l^1 norm of $\mathbf{r}$) perpendicular to e_j (since the contour must exit the excitation in the correct place), and at least $2(|I| - \sum |X_i|)$ bonds parallel or antiparallel to e_j (since I is connected and hence ω_0 must be in the excitation wherever there is no X). Thus the number of space bonds which ω_0 contributes to the sum is bounded below by $2(|I| - \sum |X_i|) + |\mathbf{r}|$. Therefore we have the bound

$$|\mathscr{U}(I, \mathbf{r}, s)| \leqslant O\left(\frac{1}{N} \exp\left[-(|I| + |\mathbf{r}|)\mu - \frac{|s|}{N}\right]\right) \tag{31}$$

To handle the constraints imposed by periodicity, we use the following identity to transform the $\delta(\sum_i (\mathbf{r}_i, s_i))\, \delta(\sum_i s_i + \sum_i t_i)$ term:

$$\delta\left(\sum_i \mathbf{r}_i\right) \delta\left(\sum_i s_i + \sum_i t_i\right)$$
$$= \frac{1}{(2\pi)^{d-1}} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_{d-1} \exp\left(i\mathbf{k} \cdot \sum_j \mathbf{r}_j\right)$$
$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \exp\left[iq\left(\sum_k s_k + \sum_l t_l\right)\right] \tag{32}$$

With the above transformation we define the weights in Fourier space by

$$\bar{\mathscr{U}}(I, \mathbf{k}, q) = \sum_{\mathbf{r}} \sum_{s} \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(iqs)\, \mathscr{U}(I, \mathbf{r}, s) \tag{33}$$

From (31) we get the bound on $\bar{\mathscr{W}}$:

$$|\bar{\mathscr{W}}(I, \mathbf{k}, q)| \leqslant \frac{1}{N} \sum_{\mathbf{r}} \sum_{s} c \exp\left[-(|I| + |\mathbf{r}|)\mu - \frac{|s|}{N} \right]$$
$$\leqslant O(\exp(-|I|\mu)) \tag{34}$$

Every site not in $\bigcup_i I$ in the expression (30) will have a factor of

$$\sum_t W_t(i) \exp(iqt) = \sum_t \frac{1}{N} \exp\left(\frac{-2|t|}{N} + \frac{iqt}{N} \right)$$
$$= \frac{1}{N}\left[1 + \frac{1}{e^{(-2+iq)/N} - 1} + \frac{1}{e^{(-2-iq)/N} - 1} \right]$$
$$\xrightarrow{N \to \infty} \frac{1}{1 + q^2/4} \tag{35}$$

Therefore we have (after using the last expression in its limiting form)

$$f^L_{\mathscr{R}_1} = \varepsilon^L d \frac{1}{(2\pi)^{d-1}} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_{d-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} dq\, Z^1(\mathbf{k}, q) \tag{36}$$

where

$$Z^1(\mathbf{k}, q) = \lim_{\beta \to \infty} \lim_{N \to \infty} \sum_{\substack{I_1, \ldots, I_m: \\ \text{disjoint}}} (1 + q^2/4)^{\sum_i |I_i| - L} \prod_i \bar{\mathscr{W}}(I_i, \mathbf{k}, q) \tag{37}$$

2.4. Analysis of the Expansion

The last equation, (37), looks like the partition function of a one-dimensional gas of interacting rods. Our next step is to show that the logarithm of the partition function of this gas, $Z^1(\mathbf{k}, q)$, has an expansion. We can use the bound from (34), but there is now a factor of $(1 + q^2/4)^{|I|}$ multiplying each of the $\bar{\mathscr{W}}(I_i, \mathbf{k}, q)$ weights. Since $|q| \leqslant \pi$, we can still obtain the bound

$$|(1 + q^2/4)^{|I|}\, \bar{\mathscr{W}}(I, \mathbf{k}, q)| \leqslant (1 + \pi^2/4)^{|I|}\, O(\exp(-|I|\mu/2)) \tag{38}$$

which is exponentially decreasing in $|I|$ for large enough μ (which can be obtained by taking small enough ε). Therefore a convergent polymer expansion exists for $\ln Z^1(\mathbf{k}, q)$:

$$Z^1(\mathbf{k}, q) = \lim_{\beta \to \infty} \lim_{N \to \infty} \left(\frac{1}{1 + q^2/4} \right)^L \exp[L g(\mathbf{k}, q) + h(\mathbf{k}, q)]$$
$$= \exp\{L[-\ln(1 + q^2/4) + g(\mathbf{k}, q)] + h(\mathbf{k}, q)\} \tag{39}$$

where $g(\mathbf{k}, g)$ is the free energy per site of the one-dimensional gas, and $h(\mathbf{k}, q)$ results from terms which go all the way from 0 to L. Using (38), it is easy to check that both h and g are uniformly bounded in $\mathbf{k}$, and g is bounded by $O(e^{-\rho L})$ [for the appropriate ρ which can be found from (38)]. We now must discuss the limits $N \to \infty$ and $\beta \to \infty$. All of the bounds which we need in order to show convergence of both expansions leading to (39) are uniform in N and β and the limits can be moved under the integrals by application of the Lebesgue dominated convergence theorem; that is, we can apply the limits to the quantities $g(\mathbf{k}, q)$ and $h(\mathbf{k}, q)$ before any integration.

To find the asymptotic behavior of $f^L_{\mathscr{R}_1}$ as $L \to \infty$, we need to find the lowest-order terms in the g and h. Of course, the terms we are looking for are in the g term, considering the bound we have on h. The lowest-order contributions to $g(\mathbf{k}, q)$ arise from those excitations which have $|I| = 1$. From that set of excitations, the contours which sidetrack only once have the greatest weights. Recall that every space bond has a weight of ε; therefore the weights of all contours other than the ones which sidetrack once are decreased by a factor of ε relative to those which sidetrack only once. In $d = 1$ contours cannot sidetrack, and so the contours with the greatest weights are those which backtrack for only one bond and then turn around into the "correct" direction again. For $d > 1$, the contours which sidetrack (from the e_1 direction) have the weight

$$\varepsilon(1 + q^2/4) \sum_{j=2}^{d} \exp(i\mathbf{k} \cdot e_j) \times \int ds_1 \int ds_2 \exp[iq(s_1 + s_2)] \exp(-2|s_1|) \exp(-2|s_2|) \tag{40}$$

After evaluating the above, we find

$$\varepsilon(1 + q^2/4)^{-1} \sum_{i=2}^{d} 2\cos(k_i) \tag{41}$$

The above expression has a maximum at $\mathbf{k} = 0$, $q = 0$. The expansion of $[-\ln(1 + q^2/4) + g(\mathbf{k}, q)]$ to orders ε, k^2, and q^2 is

$$-\varepsilon(2d-1)/4q^2 - 2\varepsilon \sum_{i=2}^{d} (1 - k_i^2)$$

Therefore, evaluating the integrals by a standard steepest descent method, we find $f^L_{\mathscr{R}_1} \sim \varepsilon^L \exp[2(d-1)L]\, L^{-d/2}$. Thus the asymptotic behavior of $|f_\infty - f_L|$ is the same, and the proof is complete.

3. CORRELATION LENGTH

Now that we have shown that the ground-state energy per site of the Ising model in a strong transverse field on a lattice of size L^d converges to its infinite-volume value as $\exp(-\xi L)/L^{d/2}$, we are able to get a result on the spin–spin correlation function of σ^x with only a bit more effort. We will show that the correlation function at zero temperature has a correlation length equal to the value of ξ in Theorem 1, and the correction to the exponential decay is asymptotically given by $l^{d/2}$. The correlation length and correction to the exponential decay of spin–spin correlations in the ground state has been found previously by Kennedy.[4]

The correlation function which we now wish to consider is $\langle \sigma_0^x \sigma_{le_i}^x \rangle$, for integer values of l. This function is given by

$$\langle \sigma_0^x \sigma_{le_i}^x \rangle = \lim_{L \to \infty} \lim_{\beta \to \infty} \mathrm{Tr}[\sigma_0^x \sigma_{le_i}^x e^{-\beta H}]/Z_L \tag{42}$$

The key to getting the result lies in the relation between the geometrical pictures of the expansions for the free energy per site and the correlation function. Using the Trotter product formula as before, we obtain the geometrical picture of this correlation function. What we find is that the spins at the origin and le_i are flipped at time 0, which yields a contour which goes between those two sites. All other contours will be closed loops as in the expansion of the free energy. The existence of expansions for both numerator and denominator of (42) follows exactly as in the previous proof. Then, after the normalization, we find that terms in the expansion consist of a single distinguished contour between $(0, 0)$ and $(le_i, 0)$ and clusters of closed loops which overlap the distinguished contour. Thus a typical term in the expansion will look something like Fig. 2. The existence

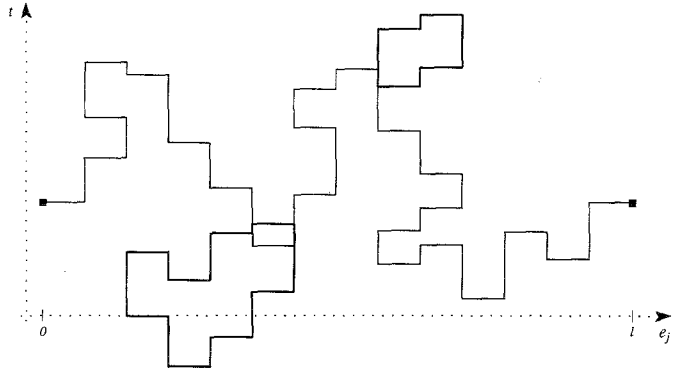

Fig. 2. A typical path in the expansion for the two-point correlation function ($d = 1$). There are excitations along the distinguished path in addition to the two closed-loop excitations overlapping the path.

of the two limits which we take is also easily established using standard results in polymer expansions.

Recall that the leading-order asymptotic behavior of the free energy per site is due to ribbons which wind around the lattice in only one of the directions. We were able to fix the direction and the location of the clusters in the lattice. For the correlation function the clusters are fixed by the location of the distinguished contour. Therefore, except for the fact that the ribbons are closed paths and the distinguished contours are not, the geometrical pictures as well as the weights are the same for the two functions. In turn, the free energies of the two gases of interacting one-dimensional rods (obtained by the second expansions for each function) are similar apart from "boundary terms." In fact the infinite-L and infinite-l free eergies per site (of the respective functions) are identical. Because the asymptotic behavior of both expansions does not depend on the boundary terms, the final result for the asymptotic behavior of both expansions is the same.

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